1.

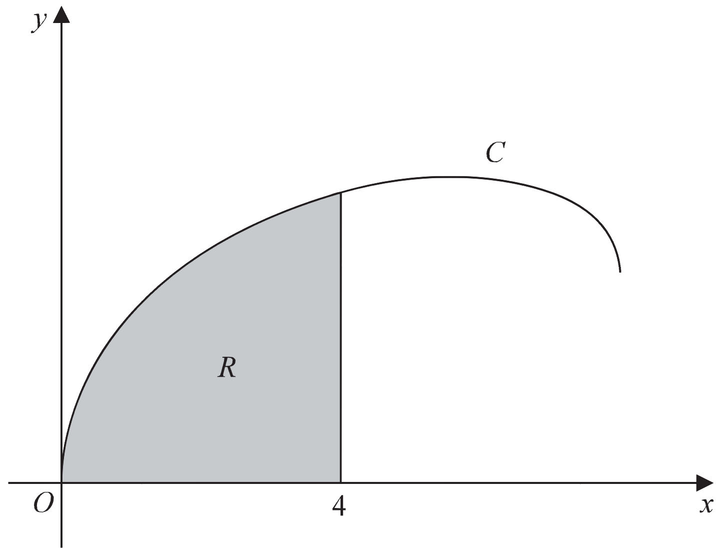

Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 8\sin^2 t \qquad y = 2\sin 2t + 3\sin t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

The region R, shown shaded in Figure 6, is bounded by C, the x-axis and the line with equation $x = 4$

(a) Show that the area of R is given by

$$\int_0^a \left(8 - 8\cos 4t + 48\sin^2 t\cos t\right)dt$$

where a is a constant to be found.

(5)

(b) Hence, using algebraic integration, find the exact area of R.

(4)

(a) $R = \int_q^p y \frac{dx}{dt}\, dt$

$x = 8\sin^2 t$

$\frac{dx}{dt} = 8 \times 2\sin t\cos t$

$= 16\sin t \cos t$

$\frac{d}{dx}\sin^2 x = 2\sin x\cos x$ using the chain rule with $u = \sin x$

$y \times \frac{dx}{dt} = (2\sin 2t + 3\sin t) \times 16\sin t\cos t$ 1

$$y \times \frac{dx}{dt} = \left[2(\sin t\cos t + \cos t \sin t) + 3\sin t\right] \times 16 \sin t \cos t$$

$$= (4\sin t\cos t + 3\sin t) \times 16 \sin t \cos t$$

$$= (64 \sin^2 t \cos t^2 + 48 \sin^2 t \cos t)$$

$$R = \int_0^a y \frac{dx}{dt}\, dt = \int_0^a 64 \sin^2 t \cos t^2 + 48 \sin^2 t \cos t \; dt \quad (1)$$

We need to find a way to simplify this into the required form.

$$\cos 4t = 2\cos^2 2t - 1$$
$$= 2(1 - \sin^2 2t) - 1$$
$$= 2 - 2\sin^2 2t - 1$$
$$= 1 - 2\sin^2 2t$$
$$= 1 - 2(\sin 2t \sin 2t)$$
$$= 1 - 2(2\sin t\cos t \times 2\sin t \cos t)$$
$$= 1 - 2(4\sin^2 t\cos^2 t)$$
$$= 1 - 8\sin^2 t \cos^2 t \quad (1)$$

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

using $\sin(2x) = 2\sin x\cos x$

$$8 \sin^2 t \cos^2 t = 1 - \cos 4t$$

$64\sin^2 t\cos^2 t = 8(1 - \cos 4t)$

$$R = \int_0^a 8 - 8\cos 4t + 48 \sin^2 t \cos t \; dt \quad (1)$$

$$a = \frac{\pi}{4} \quad (1)$$

Finding the new domain :

$$R = \int_0^4 y\, dx = \int y \frac{dx}{dt} \cdot dt$$

$$x = 8 \sin^2 t \longrightarrow \frac{dx}{dt} = 16 \sin t \cos t$$

when $x = 0$, $8 \sin^2 t = 0$, Hence, $t = 0$

when $x = 4$, $8 \sin^2 t = 4 \longrightarrow \sin^2 t = \frac{1}{2}$

$$t = \sin^{-1}\sqrt{\tfrac{1}{2}} = \pi/4$$

(b) $\int_0^{\frac{\pi}{4}} 8 - 8\cos 4t + 48\sin^2 t \cos t \; dt = 8t - 2\sin 4t + 16\sin^3 t$ (2)

$\left[8t - 2\sin 4t + 16\sin^3 t\right]_0^{\frac{\pi}{4}} = \left[8\left(\frac{\pi}{4}\right) - 2\sin\left(4 \times \frac{\pi}{4}\right) + 16\sin^3\left(\frac{\pi}{4}\right)\right]$

$- \left[8(0) - 2\sin(4 \times 0) + 16\sin^3(0)\right]$ (1)

$= 2\pi + 4\sqrt{2}$ (1)